On linearly related orthogonal polynomials and their functionals

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Abstract

Let $\{P_n\}$ be a sequence of polynomials orthogonal with respect a linear functional u and $\{Q_n\}$ a sequence of polynomials defined by

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x).$$

We find necessary and sufficient conditions in order to $\{Q_n\}$ be a sequence of polynomials orthogonal with respect to a linear functional v. Furthermore we prove that the relation between these linear functionals is $(x-\tilde{a})u = \lambda(x-a)v$. Even more, if u and v are linked in this way we get that $\{P_n\}$ and $\{Q_n\}$ satisfy a formula as above.

Keywords Orthogonal polynomials; Recurrence relations; Linear functionals

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1. Introduction

Let u be a linear functional defined in the linear space $\mathbb{P}$ of polynomials with complex coefficients.

The linear functional u is said to be quasi-definite if the matrix $H = (u_{i+j})_{i,j=0}^{\infty}$ associated with the moments $u_n = \langle u, x^n \rangle$, $n \in \mathbb{N} \cup \{0\}$, of the linear functional is quasi-definite, i.e., the principal submatrices $H_n = (u_{i+j})_{i,j=0}^{n}$, $n \in \mathbb{N} \cup \{0\}$, are nonsingular.

In such a situation, there exists a sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ such that

(i) $\deg P_n = n$,
(ii) $\langle u, P_n P_m \rangle = k_n \delta_{n,m}$ with $k_n \neq 0$.

The sequence $\{P_n\}_{n\geqslant 0}$ is said to be a sequence of monic orthogonal polynomials (SMOP) with respect to the linear functional u.

The sequence $\{P_n\}_{n\geqslant 0}$ satisfies a three-term recurrence relation of the form $xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$, $n \geqslant 0$, $\gamma_n \neq 0$, $P_{-1}(x) = 0$, $P_0(x) = 1$. Conversely, if a sequence of monic polynomials satisfies a three-term recurrence relation as above, then there exists a quasi-definite linear functional u such that $\{P_n\}_{n\geqslant 0}$ is the corresponding SMOP (see [1]).

For an SMOP $\{P_n\}_{n\geqslant 0}$ relative to u, let $\{P_n^{(1)}\}_{n\geqslant 0}$ be the associated SMOP of the first kind defined by

$$
\begin{aligned}
&P_{n+1}^{(1)}(x) = (x - \beta_{n+1}) P_n^{(1)}(x) - \gamma_{n+1} P_{n-1}^{(1)}(x), \quad n \geqslant 0,\\
&P_{-1}^{(1)}(x) = 0, \qquad P_0^{(1)}(x) = 1.
\end{aligned}
$$

Another important representation of $P_n^{(1)}(x)$ is (see [1, Chapter 3])

$$
P_n^{(1)}(y) = \frac{1}{u_0}\left\langle u, \frac{P_{n+1}(y) - P_{n+1}(x)}{y - x} \right\rangle.
$$

Also, let $\{P_n(x,\alpha)\}_{n\geqslant 0}$ be the co-recursive SMOP defined by

$$
\begin{aligned}
&P_{n+1}(x,\alpha) = (x - \beta_n) P_n(x,\alpha) - \gamma_n P_{n-1}(x,\alpha), \quad n \geqslant 1,\\
&P_1(x,\alpha) = P_1(x) - \alpha, \qquad P_0(x,\alpha) = 1.
\end{aligned}
$$

It is known (see [1,5]) that $P_n(x,\alpha) = P_n(x) - \alpha P_{n-1}^{(1)}(x)$.

For a linear functional u, a polynomial π, and a complex number a, let πu and $(x-a)^{-1}u$ be the linear functionals defined on $\mathbb{P}$ by

$$
\begin{aligned}
&\langle \pi u, P \rangle = \langle u, \pi P \rangle, \quad P \in \mathbb{P},\\
&\left\langle (x-a)^{-1}u, P \right\rangle = \left\langle u, \frac{P(x) - P(a)}{x - a} \right\rangle, \quad P \in \mathbb{P}.
\end{aligned}
$$

In the constructive theory of orthogonal polynomials the so-called inverse problem is considered. An inverse problem for linear functionals can be stated as follows: Given two sequences of monic polynomials $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, to find necessary and sufficient conditions in order to $\{Q_n\}_{n\geqslant 0}$ be an SMOP when $\{P_n\}_{n\geqslant 0}$ is an SMOP and they are related by

$$F(P_n, \ldots, P_{n-l}) = G(Q_n, \ldots, Q_{n-k}), \tag{1.1}$$

where F and G are fixed functions. As a next step, to find the relation between the functionals.

This kind of problems appear in several situations.

For instance, in [9], this problem is solved when (1.1) becomes

$$P_n(x) = Q_n(x) + a_n Q_{n-1}(x), \quad a_n \neq 0,\ n \geqslant 1.$$

Moreover, the relation between the linear functionals u and v associated with the sequences $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, respectively, is $v = M(x-a)u$ with a and M complex numbers (see Theorem 1 in [9]). This kind of transform for linear functionals is known in the literature as Christoffel transform (see [10]) or Darboux transform without free parameter for the Jacobi matrices associated with the corresponding SMOP (see [2]). In the same paper, Marcellán and Petronilho solve the inverse problem in the particular case,

$$P_n(x) + a_n P_{n-1}(x) = Q_n(x), \quad a_n \neq 0,\ n \geqslant 1.$$

In such a case, the relation satisfied by the functionals is $v = v_0\delta_a + M(x-a)^{-1}u$, where a and M are complex numbers. This kind of transform is known in the literature as Geronimus transform (see [10]) or Darboux transform with a free parameter for tridiagonal matrices in the same sense as in a previous sentence (see [2]).

In [3], the authors study when some linear combinations of two sequences of orthogonal polynomials are again orthogonal polynomial sequences. In this context these sequences are related by (1.1) with F and G linear functions. More recently, in [4], similar questions are analyzed in the framework of Sobolev inner products when one of the measures is a classical one (Hermite, Laguerre, Jacobi, Bessel).

Finally, in the framework of orthogonal polynomials with respect to measures supported on the unit circle, some inverse problems related to ARMA process have been solved in [8].

The aim of our contribution is the analysis of the following inverse problem: Given an SMOP $\{P_n\}_{n\geqslant 0}$, orthogonal with respect to a linear functional u, to find necessary and sufficient conditions in order to a sequence of monic polynomials $\{Q_n\}_{n\geqslant 0}$, defined by

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x), \quad n \geqslant 0,$$

be an SMOP with respect to a quasi-definite linear functional v. As a next step, to find the relation between the linear functionals u and v.

Another problem studied in the theory of orthogonal polynomials is the following: Given two quasi-definite linear functionals u, v such that $v = F(u)$, where F is a function in $\mathbb{P}'$, the dual space of $\mathbb{P}$, to find the explicit relations between the corresponding SMOP.

In particular, it can be shown that if $(x-a)u = \lambda v$ $(a, \lambda \in \mathbb{C})$ then $P_n(x) = Q_n(x) + a_n Q_{n-1}(x)$, $n \geqslant 0$ with $a_n \neq 0$ (see [1, Chapter 1]).

In this paper we study this problem when the linear functionals are related by the formula $(x-\tilde{a})u = \lambda(x-a)v$ $(a, \tilde{a}, \lambda \in \mathbb{C})$, which appears in the analysis of our inverse problem.

2. Main results

Lemma 2.1. *Let* $\{P_n\}_{n\geqslant 0}$ *and* $\{Q_n\}_{n\geqslant 0}$ *be sequences of monic polynomials orthogonal with respect to quasi-definite linear functionals* u *and* v*, normalized by* $\langle u, 1\rangle = 1 = \langle v, 1\rangle$*, respectively. Assume that there exist sequences of complex numbers* $\{s_n\}_{n\geqslant 1}$, $\{t_n\}_{n\geqslant 1}$ *such that the relation*

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x) \tag{2.1}$$

holds for every $n \geqslant 1$*. Thus*

(i) *If* $s_1 = t_1$*, then* $s_n = t_n$ *for every* $n \geqslant 2$*;*
(ii) *If* $s_1 \neq t_1$ *and* $s_2 = 0$*, then* $s_n = 0 \neq t_n$ *for every* $n \geqslant 2$*;*
(iii) *If* $s_1 \neq t_1$ *and* $t_2 = 0$*, then* $t_n = 0 \neq s_n$ *for every* $n \geqslant 2$*;*
(iv) *If* $s_1 \neq t_1$ *and* $s_2 t_2 \neq 0$*, then* $s_n t_n \neq 0$ *for every* $n \geqslant 2$*.*

Proof. From (2.1) it follows that

$$\langle u, Q_n\rangle = -t_n \langle u, Q_{n-1}\rangle, \quad n \geqslant 2, \qquad \langle u, Q_1\rangle = s_1 - t_1, \tag{2.2}$$

and

$$\langle v, P_n\rangle = -s_n \langle v, P_{n-1}\rangle, \quad n \geqslant 2, \qquad \langle v, P_1\rangle = t_1 - s_1. \tag{2.3}$$

If $s_1 = t_1$, either (2.2) or (2.3) yields $P_n = Q_n$ for every n and taking into account (2.1), $s_n = t_n$ for every n.

If $s_1 \neq t_1$ and $s_2 = 0$, then from (2.3) we deduce $\langle v, P_n\rangle = 0$, for every $n \geqslant 2$, and $\langle v, P_1\rangle \neq 0$. Hence, we get $P_n(x) = Q_n(x) + a_n Q_{n-1}(x)$ with $a_n \neq 0$ for every $n \geqslant 1$ (see [7]).

Substituting this relation in (2.1) we get

$$(a_n + s_n) Q_{n-1}(x) + s_n a_{n-1} Q_{n-2}(x) = t_n Q_{n-1}(x), \quad n \geqslant 1,$$

which yields $a_n + s_n = t_n$ for $n \geqslant 1$ and $s_n a_{n-1} = 0$ for $n \geqslant 2$. Then (ii) holds.

Case (iii) can be proved in the same way.

Finally, let $s_1 \neq t_1$ and $s_2 t_2 \neq 0$ and assume $s_n t_n = 0$ for some nonnegative integer $n \geqslant 3$. Write $n_0 = \min\{n \in \mathbb{N};\ n \geqslant 3,\ s_n t_n = 0\}$.

If $s_{n_0} = 0$ (the case $t_{n_0} = 0$ is analogous), then from (2.3) we deduce $\langle v, P_n\rangle = 0$ for $n \geqslant n_0$ and $\langle v, P_n\rangle \neq 0$ for $1 \leqslant n \leqslant n_0 - 1$. Hence $P_n(x) = Q_n(x) + \sum_{j=1}^{n_0-1} a_n^{(j)} Q_{n-j}(x)$ holds for every $n \geqslant n_0 - 1$, with $a_n^{(n_0-1)} \neq 0$ (see [6,7]). In the same way as in (ii), we obtain $s_{n_0-1} a_{n_0}^{(n_0-1)} = 0$, which is not possible. So, $s_n t_n \neq 0$ for $n \geqslant 3$ and (iv) follows. □

Remark. The first situation is the trivial case, i.e., $P_n = Q_n$ for every $n \geqslant 1$. The second and the third cases correspond to relations which had already been studied in [9]. For this reason, from now on, we will only consider relations like (2.1) where all the parameters do not vanish. Observe that, without lost of generality, we can suppose that $s_1 t_1 \neq 0$.

In the sequel $\{P_n\}_{n\geqslant 0}$ denotes an SMOP which satisfies the three-term recurrence relation

$$
\begin{aligned}
&P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geqslant 1, \\
&P_0(x) = 1, \qquad P_1(x) = x - \beta_0,
\end{aligned} \tag{2.4}
$$

where $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_n\}_{n\geqslant 1}$ are sequences of complex numbers with $\gamma_n \neq 0$ for $n \geqslant 1$.

Now, we characterize the orthogonality of a sequence $\{Q_n\}_{n\geqslant 0}$ of monic polynomials defined by (2.1) from an SMOP $\{P_n\}_{n\geqslant 0}$.

Theorem 2.2. *Let $\{P_n\}_{n\geqslant 0}$ be a sequence of monic orthogonal polynomials with recurrence coefficients β_n and γ_n. We define recursively a sequence $\{Q_n\}_{n\geqslant 0}$ of monic polynomials by formula* (2.1), *i.e.,*

$$
P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x), \quad n \geqslant 1,
$$

where s_n and t_n are complex numbers with $s_1 \neq t_1$ and $s_n t_n \neq 0$ for all $n \geqslant 1$. Then $\{Q_n\}_{n\geqslant 0}$ is an SMOP with recurrence coefficients $\{\tilde{\beta}_n, \tilde{\gamma}_n\}$ if and only if there exist two complex numbers a and $\tilde{a}$ such that the following formulas hold:

$$
\tilde{\gamma}_1 \neq 0, \tag{2.5}
$$

$$
\begin{aligned}
&s_2\gamma_1 - s_1\big[\gamma_2 + s_2(s_3 - s_2 - \beta_2 + \beta_1)\big] \\
&\quad = t_2\tilde{\gamma}_1 - t_1\big[\tilde{\gamma}_2 + t_2(t_3 - t_2 - \tilde{\beta}_2 + \tilde{\beta}_1)\big],
\end{aligned} \tag{2.6}
$$

$$
\beta_n - s_{n+1} - \frac{\gamma_n}{s_n} = a, \quad n \geqslant 2, \tag{2.7}
$$

$$
\tilde{\beta}_n - t_{n+1} - \frac{\tilde{\gamma}_n}{t_n} = \tilde{a}, \quad n \geqslant 2, \tag{2.8}
$$

where the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ are defined by

$$
\tilde{\beta}_n = t_{n+1} - t_n - (s_{n+1} - s_n - \beta_n), \quad n \geqslant 0, \tag{2.9}
$$

$$
\begin{aligned}
&\tilde{\gamma}_n = \gamma_n + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1}) - t_n(t_{n+1} - t_n - \tilde{\beta}_n + \tilde{\beta}_{n-1}), \\
&\quad n \geqslant 0,
\end{aligned} \tag{2.10}
$$

with $s_0 = t_0 = 0 = \gamma_0 = \tilde{\gamma}_0$.

Proof. From the definition of Q_n we get

$$
Q_{n+1}(x) = P_{n+1}(x) + s_{n+1}P_n(x) - t_{n+1}Q_n(x), \quad n \geqslant 0. \tag{2.11}
$$

Inserting formula (2.4) in (2.11) and applying (2.1) to $xP_n(x)$, we get that

$$
\begin{aligned}
Q_{n+1}(x) = {} & xQ_n(x) + (s_{n+1} - s_n - \beta_n)P_n(x) + t_n x Q_{n-1}(x) - t_{n+1}Q_n(x) \\
& - (s_n\beta_{n-1} + \gamma_n)P_{n-1}(x) - s_n\gamma_{n-1}P_{n-2}(x), \quad n \geqslant 1,
\end{aligned}
$$

follows, provided we substitute there $xP_{n-1}(x)$, using again (2.4). Now, formula (2.1) applied to $P_n(x)$ and the definition of $\tilde{\beta}_n$ (see (2.9)), yield

$$
\begin{aligned}
Q_{n+1}(x) = (x-\tilde{\beta}_n)Q_n(x) &+ t_n(t_{n+1}-t_n-\tilde{\beta}_n)Q_{n-1}(x)\\
&- \big[s_n(s_{n+1}-s_n-\beta_n+\beta_{n-1})+\gamma_n\big]P_{n-1}(x) - s_n\gamma_{n-1}P_{n-2}(x)\\
&- t_n\big[Q_n(x)-xQ_{n-1}(x)\big]
\end{aligned}
$$

for $n \geqslant 0$. So $\{Q_n\}_{n\geqslant 0}$ is an SMOP if and only if there exists a sequence of complex numbers $(\tilde{\gamma}_n)_1^\infty$ with $\tilde{\gamma}_n \neq 0$ for $n \geqslant 1$, such that

$$
\begin{aligned}
&t_n(t_{n+1}-t_n-\tilde{\beta}_n)Q_{n-1}(x) - \big[s_n(s_{n+1}-s_n-\beta_n+\beta_{n-1})+\gamma_n\big]P_{n-1}(x)\\
&\quad - s_n\gamma_{n-1}P_{n-2}(x) - t_n\big[Q_n(x)-xQ_{n-1}(x)\big] = -\tilde{\gamma}_n Q_{n-1}(x).
\end{aligned}
\tag{2.12}
$$

Moreover, $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ are the three-term recurrence coefficients for Q_n.

Next, we are going to see that $\{Q_n\}_{n\geqslant 0}$ is an SMOP if and only if, for every $n \geqslant 1$, the relation

$$
\begin{aligned}
&\big[\tilde{\gamma}_n + t_n(t_{n+1}-t_n-\tilde{\beta}_n+\tilde{\beta}_{n-1})\big]Q_{n-1}(x) + t_n\tilde{\gamma}_{n-1}Q_{n-2}(x)\\
&\quad = \big[\gamma_n + s_n(s_{n+1}-s_n-\beta_n+\beta_{n-1})\big]P_{n-1}(x) + s_n\gamma_{n-1}P_{n-2}(x)
\end{aligned}
\tag{2.13}
$$

holds, where $\tilde{\gamma}_n$ is given by (2.10).

Suppose that $\{Q_n\}_{n\geqslant 0}$ is an SMOP. Then, it is enough to substitute the expression $Q_n(x) - xQ_{n-1}(x)$ from the three-term recurrence relation in formula (2.12) to obtain (2.13).

Conversely, if (2.13) is satisfied then we show that the sequence $\{Q_n\}_{n\geqslant 0}$ satisfies a three-term recurrence relation, that is, $\{Q_n\}_n \geqslant 0$ is an SMOP.

Indeed, applying (2.4) in (2.13), and the definition of $\tilde{\beta}_n$, for $n \geqslant 1$ we get

$$
\begin{aligned}
&t_n\big(\tilde{\beta}_{n-1}Q_{n-1}(x) + \tilde{\gamma}_{n-1}Q_{n-2}(x)\big)\\
&\quad = \gamma_n P_{n-1}(x) + (t_{n+1}-t_n-\tilde{\beta}_n)\big[s_nP_{n-1}(x) - t_nQ_{n-1}(x)\big]\\
&\qquad + s_n\big[xP_{n-1}(x) - P_n(x)\big] - \tilde{\gamma}_nQ_{n-1}(x).
\end{aligned}
$$

Substituting (2.1) in $s_nP_{n-1}(x) - t_nQ_{n-1}(x)$ and, using again the definition of $\tilde{\beta}_n$, for $n \geqslant 1$ we have

$$
\begin{aligned}
&t_n\big(\tilde{\beta}_{n-1}Q_{n-1}(x) + \tilde{\gamma}_{n-1}Q_{n-2}(x)\big)\\
&\quad = \gamma_n P_{n-1}(x) + (t_{n+1}-t_n-\tilde{\beta}_n)Q_n(x) - (s_{n+1}-\beta_n)P_n(x)\\
&\qquad + s_nxP_{n-1}(x) - \tilde{\gamma}_nQ_{n-1}(x).
\end{aligned}
$$

Applying (2.1) in $s_nP_{n-1}(x)$ as well as the recurrence relation for $\{P_n\}_{n\geqslant 0}$, we get

$$
\begin{aligned}
&t_n\big(\tilde{\beta}_{n-1}Q_{n-1}(x) + \tilde{\gamma}_{n-1}Q_{n-2}(x)\big)\\
&\quad = t_n\big[xQ_{n-1}(x) - Q_n(x)\big] - P_{n+1}(x)\\
&\qquad + t_{n+1}Q_n(x) - s_{n+1}P_n(x) - \tilde{\beta}_nQ_n(x) + xQ_n(x) - \tilde{\gamma}_nQ_{n-1}(x), \quad n \geqslant 1.
\end{aligned}
$$

Using again (2.1),

$$
\begin{aligned}
&t_n\big[Q_n(x) - (x-\tilde{\beta}_{n-1})Q_{n-1}(x) + \tilde{\gamma}_{n-1}Q_{n-2}(x)\big]\\
&\quad = -Q_{n+1}(x) + (x-\tilde{\beta}_n)Q_n(x) - \tilde{\gamma}_nQ_{n-1}(x), \quad n \geqslant 1,
\end{aligned}
$$

i.e.,

$$Q_{n+1}(x) = (x - \tilde{\beta}_n)Q_n(x) - \tilde{\gamma}_n Q_{n-1}(x), \quad n \geqslant 0. \tag{2.14}$$

Now, we will show that (2.12) is equivalent to formulas (2.6)–(2.8) in the statement of the theorem.

From (2.1) it follows that formula (2.13) is equivalent to

$$\begin{aligned}&\big\{t_n\tilde{\gamma}_{n-1} - t_{n-1}\big[\tilde{\gamma}_n + t_n(t_{n+1} - t_n - \tilde{\beta}_n + \tilde{\beta}_{n-1})\big]\big\}Q_{n-2}(x)\\ &\quad = \big\{s_n\gamma_{n-1} - s_{n-1}\big[\gamma_n + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1})\big]\big\}P_{n-2}(x)\end{aligned}$$

for every $n \geqslant 2$.

For $n = 2$, we obtain (2.6) and when $n \geqslant 3$, both coefficients in the last formula vanish. Thus

$$s_n\gamma_{n-1} = s_{n-1}\big[\gamma_n + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1})\big], \tag{2.15}$$

$$t_n\tilde{\gamma}_{n-1} = t_{n-1}\big[\tilde{\gamma}_n + t_n(t_{n+1} - t_n - \tilde{\beta}_n + \tilde{\beta}_{n-1})\big] \tag{2.16}$$

hold. As a consequence, since $s_n t_n \neq 0$ for every $n \geqslant 1$, (2.7) and (2.8) follow.

Conversely, it is easy to verify that from (2.6)–(2.8) we deduce (2.13). □

Remarks. (1) Notice that, from (2.10), (2.15), and (2.16), we have

$$\frac{t_{n+1}}{t_n}\tilde{\gamma}_n = \frac{s_{n+1}}{s_n}\gamma_n \quad \text{for every } n \geqslant 2. \tag{2.17}$$

Thus, $\tilde{\gamma}_n \neq 0$ for every $n \geqslant 2$.

(2) We want to point out that there are four initial conditions: s_1, t_1, s_2, t_2 connected among them by the condition $\tilde{\gamma}_1 \neq 0$. From the definition of $\tilde{\gamma}_2$ and formula (2.6) we get s_3, which allows us to deduce $\tilde{\gamma}_2$, and from (2.17), t_3. Finally, from (2.7) and (2.8) the values of s_n and t_n, with $n \geqslant 4$, can be obtained.

Proposition 2.3. *Let $\{P_n\}_{n\geqslant 0}$ be an SMOP and $\{s_n\}_{n\geqslant 1}, \{t_n\}_{n\geqslant 1}$ sequences of complex numbers such that $s_1 \neq t_1$ and $s_n t_n \neq 0$ for $n \geqslant 1$. If $\{Q_n\}_{n\geqslant 0}$ is a sequence of monic polynomials defined by* (2.1)*, then the orthogonality of $\{Q_n\}_{n\geqslant 0}$ depends at most of the choice of the parameters s_1, t_1, s_2, t_2. More precisely, $\{Q_n\}_{n\geqslant 0}$ is an SMOP if and only if the following conditions hold*:

(i) *The parameter $\tilde{\gamma}_1$, defined by* (2.10)*, is different from zero*;

(ii) *Formula* (2.6) *in Theorem* 2.2 *is true*;

(iii) $$S_n(a) \neq 0 \quad \textit{and} \quad s_n = \frac{-S_n(a)}{S_{n-1}(a)}, \quad n \geqslant 1,$$

where $a = \beta_2 - s_3 - \gamma_2/s_2$ and S_n is the generalized co-recursive polynomial of order 1 *with parameter μ of the co-recursive polynomial of $P_n(x,\alpha)$, being $\mu = s_2 - \beta_1 + \gamma_1/s_1 + a$ and $\alpha = s_1 + a - \beta_0$*;

(iv) $$T_n(\tilde{a}) \neq 0 \quad \textit{and} \quad t_n = \frac{-T_n(\tilde{a})}{T_{n-1}(\tilde{a})}, \quad n \geqslant 1,$$

where $\tilde{a} = \tilde{\beta}_2 - t_3 - \tilde{\gamma}_2/t_2$ and T_n is the generalized co-recursive polynomial of order 1 *with parameter $\tilde{\mu}$ of the co-recursive polynomial of $Q_n(x, \tilde{\alpha})$, being $\tilde{\mu} = t_2 - \tilde{\beta}_1 + \tilde{\gamma}_1/t_1 + \tilde{a}$ and $\tilde{\alpha} = t_1 + \tilde{a} - \tilde{\beta}_0$.*

Proof. According to Theorem 2.2 it is enough to show that the conditions (iii) and (iv) are equivalent to formulas (2.7) and (2.8).

In order to do this, we define a sequence $\{y_n\}_{n\geqslant 0}$ by $y_0 = 1$ and $y_n = -s_n y_{n-1}$ for every $n \geqslant 1$.

Thus $y_n \neq 0$ for $n \geqslant 0$ and, taking into account (2.7) in Theorem 2.2,

$$y_{n+1} = (a - \beta_n) y_n - \gamma_n y_{n-1}, \quad n \geqslant 2,$$
$$y_2 = (a - \beta_1 - \mu) y_1 - \gamma_1, \qquad y_1 = a - \beta_0 - \alpha$$

hold, with α and μ defined as above. These formulas imply that $y_n = S_n(a)$, $n \geqslant 0$ (see [5]), and therefore (iii) is true.

In a similar way, using (2.8), we conclude (iv).

Straightforward calculations allow us to deduce the converse. □

Next, we characterize when two sequences of monic orthogonal polynomials $\{P_n\}$ and $\{Q_n\}$ are related by formula (2.1), whenever all the coefficients are nonzero, in terms of their functionals.

Theorem 2.4. *Let u and v be quasi-definite linear functionals, normalized by $\langle u, 1\rangle = 1 = \langle v, 1\rangle$ and $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ their corresponding SMOP with recurrence coefficients $\{\beta_n, \gamma_n\}$ and $\{\tilde{\beta}_n, \tilde{\gamma}_n\}$, respectively. Then, the following conditions are equivalent*:

(i) *There exist complex sequences $\{s_n\}_{n\geqslant 1}$, $\{t_n\}_{n\geqslant 1}$ with $s_1 \neq t_1$ and $s_n t_n \neq 0$ for $n \geqslant 1$, such that $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ are related by* (2.1), *i.e.,*

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x), \quad n \geqslant 1;$$

(ii) *For every $n \geqslant 1$, $P_n \neq Q_n$ and there exist complex numbers $\lambda, a, \tilde{a}$ such that*

$$(x - \tilde{a})u = \lambda(x - a)v. \tag{2.18}$$

Moreover, for every $n \geqslant 2$, $a = \beta_n - s_{n+1} - \gamma_n/s_n$ and $\tilde{a} = \tilde{\beta}_n - t_{n+1} - \tilde{\gamma}_n/t_n$.

Proof. (i) $\Rightarrow$ (ii) From (2.3) we have $\langle v, P_n\rangle = (-1)^{n+1} s_n \dots s_2 (t_1 - s_1)$ and $\langle v, P_1\rangle = t_1 - s_1$. This implies $\langle v, P_n\rangle \neq 0$ for all $n \geqslant 1$ and then $P_n \neq Q_n$ for every $n \geqslant 1$.

Because of formula (2.1) and the orthogonality of $\{P_n\}_{n\geqslant 0}$ with respect to u, for every $A \in \mathbb{C}$, by straightforward calculations, we get

$$\left\langle (x + A)u, Q_2 \right\rangle = (s_2 - t_2)\gamma_1 - t_2(s_1 - t_1)(\beta_0 + A). \tag{2.19}$$

If we choose

$$A = \frac{\gamma_1(s_2 - t_2)}{t_2(s_1 - t_1)} - \beta_0,$$

then we have $\langle (x + A)u, Q_2 \rangle = 0$. From this, using again (2.1), by induction we obtain that $\langle (x + A)u, Q_n \rangle = 0$ for $n \geqslant 2$. So, if we expand $(x + A)u$ in the dual basis $\{Q_j v / \langle v, Q_j^2 \rangle\}_{j \geqslant 0}$ (see [7]), it follows that

$$(x + A)u = \sum_{j=0}^{1} \mu_j \frac{Q_j v}{\langle v, Q_j^2 \rangle}, \tag{2.20}$$

where

$$\mu_0 = (\beta_0 + A) = \frac{\gamma_1 (s_2 - t_2)}{t_2 (s_1 - t_1)} \quad \text{and} \quad \mu_1 = \big[\gamma_1 + (s_1 - t_1)(\beta_0 + A)\big] = \frac{\gamma_1 s_2}{t_2}.$$

In other words,

$$\left[x - \beta_0 + \frac{\gamma_1 (s_2 - t_2)}{t_2 (s_1 - t_1)}\right] u = \frac{\gamma_1 s_2}{\tilde{\gamma}_1 t_2} \left[x - \tilde{\beta}_0 + \frac{\tilde{\gamma}_1 (s_2 - t_2)}{s_2 (s_1 - t_1)}\right] v. \tag{2.21}$$

From (2.9) and (2.10), written for $n = 1$, it follows that

$$\tilde{\gamma}_1 = \gamma_1 + (s_1 - t_1)(s_2 - s_1 - \beta_1) + s_1 \beta_0 - t_1 \tilde{\beta}_0 = \gamma_1 + (s_1 - t_1)(s_2 - \beta_1 + \tilde{\beta}_0),$$

where we have used that $s_1 - \beta_0 = t_1 - \tilde{\beta}_0$.

Hence, we get

$$\frac{\tilde{\gamma}_1 (s_2 - t_2)}{s_2 (s_1 - t_1)} = \frac{\gamma_1 s_2 - \tilde{\gamma}_1 t_2}{s_2 (s_1 - t_1)} + s_2 - \beta_1 + \tilde{\beta}_0. \tag{2.22}$$

On the other hand, using (2.10) and (2.7) written for $n = 2$, and (2.6), we obtain

$$\frac{\gamma_1 s_2 - \tilde{\gamma}_1 t_2}{s_2 (s_1 - t_1)} = \frac{\gamma_2}{s_2} + s_3 - s_2 - \beta_2 + \beta_1 = -s_2 + \beta_1 - a. \tag{2.23}$$

So, (2.22) and (2.23) lead to

$$-\tilde{\beta}_0 + \frac{\tilde{\gamma}_1 (s_2 - t_2)}{s_2 (s_1 - t_1)} = -a.$$

In a similar way, it can be proved that

$$-\beta_0 + \frac{\gamma_1 (s_2 - t_2)}{t_2 (s_1 - t_1)} = -\tilde{a}.$$

Therefore relation (2.18) for the linear functionals u and v follows from (2.21).

(ii) $\Rightarrow$ (i) Suppose that the linear functionals u, v satisfy (2.18). Consider the Fourier expansion of P_n in terms of the polynomials Q_n, that is, $P_n(x) = Q_n(x) + \sum_{j=0}^{n-1} \lambda_{nj} Q_j(x)$, where $\lambda_{nj} = \langle v, P_n Q_j \rangle / \langle v, Q_j^2 \rangle$.

Since

$$v = \frac{1}{\lambda}\big(1 + (a - \tilde{a})(x - a)^{-1}\big) u + \frac{\lambda - 1}{\lambda} \delta_a,$$

we get, for $0 \leqslant j \leqslant n - 1$,

$$\langle v, P_n Q_j \rangle = \frac{a-\tilde{a}}{\lambda}\left\langle u, \frac{P_n(x)Q_j(x) - P_n(a)Q_j(a)}{x-a}\right\rangle + \frac{\lambda-1}{\lambda} P_n(a)Q_j(a)$$
$$= \frac{a-\tilde{a}}{\lambda}\left[\left\langle u, \frac{Q_j(x)-Q_j(a)}{x-a} P_n(x)\right\rangle + Q_j(a)\left\langle u, \frac{P_n(x)-P_n(a)}{x-a}\right\rangle\right]$$
$$+ \frac{\lambda-1}{\lambda} P_n(a)Q_j(a) = \frac{1}{\lambda} Q_j(a)\big[(\lambda-1)P_n(a) + (a-\tilde{a})P_{n-1}^{(1)}(a)\big],$$

where $\{P_n^{(1)}\}_{n\geqslant 0}$ denotes the sequence of associated polynomials of first kind for the SMOP $\{P_n\}_{n\geqslant 0}$. Then

$$P_n(x) = Q_n(x) + \frac{1}{\lambda}\big[(\lambda-1)P_n(a) + (a-\tilde{a})P_{n-1}^{(1)}(a)\big]K_{n-1}(x,a;v), \quad n \geqslant 1, \tag{2.24}$$

and

$$\langle v, P_n\rangle = \frac{1}{\lambda}\big[(\lambda-1)P_n(a) + (a-\tilde{a})P_{n-1}^{(1)}(a)\big], \quad n\geqslant 1, \tag{2.25}$$

where $K_{n-1}(x,a;v)$ denotes the usual reproducing kernel associated with v.

In a similar way, we get

$$Q_n(x) = P_n(x) + \big[(1-\lambda)Q_n(\tilde{a}) + \lambda(\tilde{a}-a)Q_{n-1}^{(1)}(\tilde{a})\big]K_{n-1}(x,\tilde{a};u), \quad n\geqslant 1,$$

and

$$\langle u, Q_n\rangle = \big[(1-\lambda)Q_n(\tilde{a}) + \lambda(\tilde{a}-a)Q_{n-1}^{(1)}(\tilde{a})\big],$$

where $K_{n-1}(x,\tilde{a};u)$ and $\{Q_n^{(1)}\}_{n\geqslant 0}$ denote the reproducing kernel associated with u and the sequence of associated polynomials of first kind for the SMOP $\{Q_n\}_{n\geqslant 0}$, respectively.

Observe that from the condition $P_n \neq Q_n$, $n\geqslant 1$, we get $\langle v, P_n\rangle \neq 0$ and $\langle u, Q_n\rangle \neq 0$ for all $n\geqslant 1$. Then, writing formula (2.24) for n and $n-1$, easy computations yield

$$P_n(x) - \frac{\langle v, P_n\rangle}{\langle v, P_{n-1}\rangle} P_{n-1}(x) = Q_n(x) - \left[\frac{\langle v, P_n\rangle}{\langle v, P_{n-1}\rangle} - \frac{\langle v, P_n\rangle Q_{n-1}(a)}{\langle v, Q_{n-1}^2\rangle}\right] Q_{n-1}(x)$$

for every $n\geqslant 2$. Now, applying the linear functional u we get

$$\frac{\langle u, Q_n\rangle}{\langle u, Q_{n-1}\rangle} = \left[\frac{\langle v, P_n\rangle}{\langle v, P_{n-1}\rangle} - \frac{\langle v, P_n\rangle Q_{n-1}(a)}{\langle v, Q_{n-1}^2\rangle}\right],$$

that is,

$$P_n(x) - \frac{\langle v, P_n\rangle}{\langle v, P_{n-1}\rangle} P_{n-1}(x) = Q_n(x) - \frac{\langle u, Q_n\rangle}{\langle u, Q_{n-1}\rangle} Q_{n-1}(x), \quad n\geqslant 2,$$

and therefore (i) holds with $s_n = -\langle v, P_n\rangle/\langle v, P_{n-1}\rangle \neq 0$ and $t_n = -\langle u, Q_n\rangle/\langle u, Q_{n-1}\rangle \neq 0$ for every $n\geqslant 2$.

Since $P_1 \neq Q_1$ we can write $P_1(x) + s_1 = Q_1(x) + t_1$ with $s_1 \neq t_1$ and $s_1 t_1 \neq 0$.

Finally, from (2.25) we have that $\langle v, P_n\rangle$, up to a constant factor, is the evaluation in a of some orthogonal polynomial (either P_n or $P_{n-1}^{(1)}$ or the co-recursive polynomial of P_n). Thus, $\langle v, P_{n+1}\rangle = (a-\beta_n)\langle v, P_n\rangle - \gamma_n\langle v, P_{n-1}\rangle$ for $n\geqslant 2$ and therefore $a = \beta_n - s_{n+1} - \gamma_n/s_n$ for $n\geqslant 2$. In a similar way, taking into account the explicit expression of $\langle u, Q_n\rangle$ for all $n\geqslant 1$, we obtain $\tilde{a} = \tilde{\beta}_n - t_{n+1} - \tilde{\gamma}_n/t_n$ for $n\geqslant 2$. $\square$

Remarks. (1) In the second part of the proof (that is (ii) $\Rightarrow$ (i)), the condition $P_n \neq Q_n$ for each n is necessary. Indeed, if (i) is true and $P_n = Q_n$ for some $n \geqslant 2$, then $s_n = t_n$ and $P_{n-1} = Q_{n-1}$; thus $\langle v, P_n\rangle = 0 = \langle v, P_{n-1}\rangle$ which is not possible since $\langle v, P_n\rangle = cR_n(a)$, where $c \neq 0$ and $\{R_n(x)\}_{n\geqslant 0}$ is a sequence of orthogonal polynomials.

(2) In general (2.18) does not imply $P_n \neq Q_n$ for each n. It is enough to take $a = 0 = \tilde{a}$ and v the Hermite linear functional. In such a case it can be shown that $P_{2n-1} = Q_{2n-1}$ for every $n \geqslant 1$.

On the other hand, under the conditions of Theorem 2.4 we have seen that there exists a complex number a such that we have $P_n(x) - Q_n(x) = \langle v, P_n\rangle K_{n-1}(x, a; v)$, $n \geqslant 1$, and therefore

$$t_n - s_n = \langle v, P_n\rangle \frac{Q_{n-1}(a)}{\langle v, Q_{n-1}^2\rangle}, \quad n \geqslant 1.$$

Since $\langle v, P_n\rangle \neq 0$, $n \geqslant 1$, it follows that for every $n \geqslant 1$,

$$Q_{n-1}(a) \neq 0 \quad \Leftrightarrow \quad t_n \neq s_n.$$

Analogously, for every $n \geqslant 1$,

$$P_{n-1}(\tilde{a}) \neq 0 \quad \Leftrightarrow \quad t_n \neq s_n.$$

That is, both linear functionals $(x - a)v$ and $(x - \tilde{a})u$ are quasi-definite if and only if for every $n \geqslant 1$, $t_n \neq s_n$.

We can obtain a more simple expression for the parameters s_n and t_n when the linear functional $(x - \tilde{a})u$ is quasi-definite. Actually, let $\{W_n\}_{n\geqslant 0}$ be the SMOP with respect to the quasi-definite linear functional $w = (x - \tilde{a})u = \lambda(x - a)v$. By Theorem 1 in [9] we get

$$P_n(x) = W_n(x) - a_{n-1}W_{n-1}(x), \quad n \geqslant 1, \tag{2.26}$$

$$Q_n(x) = W_n(x) - b_{n-1}W_{n-1}(x), \quad n \geqslant 1, \tag{2.27}$$

with

$$a_{n-1} = \gamma_n \frac{P_{n-1}(\tilde{a})}{P_n(\tilde{a})} \neq 0 \quad \text{and} \quad b_{n-1} = \tilde{\gamma}_n \frac{Q_{n-1}(a)}{Q_n(a)} \neq 0, \quad n \geqslant 1.$$

Observe that $P_1(x) - b_0 = Q_1(x) - a_0$.

Now, suppose $n \geqslant 2$. From (2.26) and (2.27) written for n and $n - 1$ we deduce that

$$\begin{vmatrix} 1 & -a_{n-1} & 0 & P_n(x) \\ 0 & 1 & -a_{n-2} & P_{n-1}(x) \\ 1 & -b_{n-1} & 0 & Q_n(x) \\ 0 & 1 & -b_{n-2} & Q_{n-1}(x) \end{vmatrix} = 0$$

and so $P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x)$, $n \geqslant 2$, with

$$s_n = -b_{n-2}\frac{a_{n-1} - b_{n-1}}{a_{n-2} - b_{n-2}} \quad \text{and} \quad t_n = -a_{n-2}\frac{a_{n-1} - b_{n-1}}{a_{n-2} - b_{n-2}}. \tag{2.28}$$

Notice that, since $P_n \neq Q_n$ for every $n \geqslant 1$, we have $a_{n-1} \neq b_{n-1}$, $n \geqslant 1$.

Moreover, it also follows that

$$W_n(x) = (b_{n-1} - a_{n-1})^{-1}\big(P_n(x) - Q_n(x)\big) = (t_n - s_n)^{-1}\big(P_n(x) - Q_n(x)\big).$$

In order to illustrate the results of Theorem 2.4 we show an example providing a relation for Jacobi polynomials which, as far as we know, is new.

Assume $\alpha, \beta > 0$. Let u and v be the Jacobi linear functionals with parameters $\alpha - 1, \beta$ and $\alpha, \beta - 1$, respectively, normalized by $\langle u, 1\rangle = 1 = \langle v, 1\rangle$. Denote by $P_n^{(\alpha-1,\beta)}$ and $P_n^{(\alpha,\beta-1)}$ the corresponding sequences of monic orthogonal polynomials.

Since $(1-x)u = \alpha\beta^{-1}(1+x)v$ and the linear functional $(1-x)u$ is quasi-definite, from (2.26) and (2.27) we have

$$a_n = \gamma_{n+1}^{(\alpha-1,\beta)} \frac{P_n^{(\alpha-1,\beta)}(1)}{P_{n+1}^{(\alpha-1,\beta)}(1)} \quad \text{and} \quad b_n = \gamma_{n+1}^{(\alpha,\beta-1)} \frac{P_n^{(\alpha,\beta-1)}(-1)}{P_{n+1}^{(\alpha,\beta-1)}(-1)}.$$

Using the properties of monic Jacobi polynomials (see [1]) and formula (2.28) we can obtain

$$\begin{aligned} &P_n^{(\alpha-1,\beta)}(x) + \frac{2n(n+\alpha-1)}{(2n+\alpha+\beta-2)(2n+\alpha+\beta-1)} P_{n-1}^{(\alpha-1,\beta)}(x) \\ &\quad = P_n^{(\alpha,\beta-1)}(x) - \frac{2n(n+\beta-1)}{(2n+\alpha+\beta-2)(2n+\alpha+\beta-1)} P_{n-1}^{(\alpha,\beta-1)}(x). \end{aligned} \tag{2.29}$$

For Jacobi polynomials $p_n^{(\alpha,\beta)}$ with the classical normalization $p_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$, we have the more simple relation

$$\begin{aligned} &p_n^{(\alpha-1,\beta)}(x) + \frac{n+\alpha-1}{n+\alpha+\beta-1} p_{n-1}^{(\alpha-1,\beta)}(x) \\ &\quad = p_n^{(\alpha,\beta-1)}(x) - \frac{n+\beta-1}{n+\alpha+\beta-1} p_{n-1}^{(\alpha,\beta-1)}(x). \end{aligned}$$

Relation (2.29) can be also obtained via the Darboux transformation without free parameter. Indeed, the sequences $\{P_n^{(\alpha-1,\beta)}\}_{n\geqslant 0}$ and $\{P_n^{(\alpha,\beta-1)}\}_{n\geqslant 0}$ are both (different) Darboux transforms of the sequence $\{P_n^{(\alpha-1,\beta-1)}\}_{n\geqslant 0}$. Explicitly, see [1, Exercise 7.8, p. 39],

$$\begin{aligned} &(x+1)P_n^{(\alpha-1,\beta)}(x) \\ &\quad = \frac{2n(n+\beta)(n+\alpha+\beta-1)}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)} P_n^{(\alpha-1,\beta-1)}(x) + P_{n+1}^{(\alpha-1,\beta-1)}(x), \\ &(x-1)P_n^{(\alpha,\beta-1)}(x) \\ &\quad = -\frac{2n(n+\alpha)(n+\alpha+\beta-1)}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)} P_n^{(\alpha-1,\beta-1)}(x) + P_{n+1}^{(\alpha-1,\beta-1)}(x). \end{aligned} \tag{2.30}$$

On the other hand, Theorem 2.4 assures that the polynomials $P_n^{(\alpha-1,\beta)}$ and $P_n^{(\alpha,\beta-1)}$ satisfy a formula of the form (2.1). Plugging the relations (2.30) into the formula (2.1) we can get (2.29).

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